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Note

A Note on the Asymptotics of Orthogonal Polynomials on a Complex Arc: The Case of a Measure with a Discrete Part

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The asymptotics of orthogonal polynomials are studied. The measure is concentrated on a complex arc and has masses in the region exterior to the arc, and we obtain strong asymptotics for the orthogonal polynomials. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper we study the asymptotics of orthogonal polynomials with respect to a measure of the type $\sigma = \alpha + \gamma$, where α is concentrated on a complex rectifiable arc E and is absolutely continuous with respect to the Lebesgue measure $|d\zeta|$ on the arc, and γ is a discrete measure with masses A_k at the points $z_k \in \text{Ext}(E)$, $k = 1, 2, ..., N$. For the case $E = [-1, 1]$ this problem is closely related to the scattering problem for the discrete version of the Schrödinger equation and Padé approximations of some Stieltjes-type functions. In this case the asymptotics of the associated orthogonal polynomials were investigated in [2, 1, 7, 6, 8] (see the survey on this subject in [9]). In general this investigation is based on the well-known three-term recurrences for orthogonal polynomials. In the case of a complex arc we do not have this recurrence and the proof for the asymptotics is no longer valid. We propose here another approach based on the extremal property of orthogonal polynomials (in the case of polynomials orthogonal to a system of complex curves and arcs, this method was developed by Widom in [10]). To state and prove the main result, Theorem 1, we use notations and lemmas from our previous papers

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[4, 3], where the case of a complex curve was studied. We note that the case of a closed curve is quite different from the case of an arc with respect to the asymptotics of orthogonal polynomials.

2. Notations and Basic Lemmas

Suppose E is a rectifiable arc in the complex plane, $\Omega = \text{Ext}(E)$, $G = \{w \in C, |w| > 1\}$, and $\Phi: \Omega \to G$ is the conformal mapping with $\Phi(\infty) = \infty$, $\lim_{z \to \infty}(\Phi(z)/z) > 0$. We note by Ψ the inverse function $\Phi: G \to \Omega$. As above, the measure σ is a sum of two measures: $\sigma = \alpha + \gamma$, with $d\alpha = \varrho(\zeta)\,|d\zeta|$, $\varrho(\zeta)$ an integrable non-negative function on E, and $\gamma = \sum_{k=1}^{N} A_k\,\delta(z - z_k)$, $z_k \in \Omega$. If the weight function $\varrho(\zeta)$ satisfies the Szegő condition

$$\int_E \log \varrho(\zeta)\,|\Phi'(\zeta)|\,|d\zeta| > -\infty, \tag{1}$$

then one can construct the so-called Szegő function $D(z)$ associated with domain Ω and weight function $\varrho(\zeta)$ with the following properties (see [10, 5]):

1. $D(z)$ is analytic in Ω, $D(z) \neq 0$ in Ω, and $D(0) > 0$;
2. $D(z)$ has boundary values on both sides of E (a.e.) and

$$|D_\pm|^{-2}\,|\Phi'_\pm(\zeta)| = \varrho(\zeta) \qquad \text{(a.e. on } E\text{)}.$$

Let $H^2(\Omega, \rho)$ be the Hardy space of functions analytic in Ω. We say that $f(z) \in H^2(\Omega, \rho)$ iff $f(z)$ is analytic in Ω and $f(\Psi(w))/D(\Psi(w))$ is a function from the usual space $H^2(G)$. Any function $f(z)$ from $H^2(\Omega, \rho)$ has boundary values f_+, f_- on both sides of E, $f_+, f_- \in L^2(\rho)$. We define the norm in Hardy space by

$$\|f\|_{H^2(\Omega,\rho)} = \oint_E |f(\zeta)|^2\,\rho(\zeta)\,|d\zeta|.$$

Here, we take the integral on both sides of E. Suppose $\rho(\zeta)$ satisfies Szegő condition (1) and define $\mu(\rho)$ as the extremal value of the following problem:

$$\mu(\varrho) := \inf\{\|\phi\|^2_{H^2(\Omega,\varrho)}, \phi \in H^2(\Omega, \varrho), \phi(\infty) = 1\}. \tag{2}$$

The extremal function ϕ^* for this problem is exactly $D(z)/D(0)$ (see [10, 3]). We denote by ψ^* the extremal function of the problem

$$\mu^*(\varrho) := \inf\{\|\phi\|^2_{H^2(\Omega,\varrho)}, \phi(\infty) = 1, \phi(z_k) = 0, k = 1, 2, ..., N\}. \tag{3}$$

LEMMA 1. *The extremal functions of problems* (2) *and* (3) *are connected by*

$$\psi^* = \phi^* B(z) = \phi^* \prod_{k=1}^{N} \frac{\Phi(z) - \Phi(z_k)}{\Phi(z)\,\overline{\Phi(z_k)} - 1} \frac{|\Phi(z_k)|^2}{\Phi(z_k)} \tag{4}$$

and

$$\mu^*(\varrho) = \left(\prod_{1}^{N} |\Phi(z_k)| \right)^2 \mu(\rho).$$

A proof for the case of a closed curve is given in [4]. This proof is based on the following properties of the Blashke product $B(z)$: $B(z_k) = 0$, $k = 1, 2, ..., N$, $B(\infty) = 1$, $B(z)$ has boundary values on E, and $|B(\zeta)| = \text{const} = \prod |\Phi(z_k)|$. $B(z)$ has the same properties for an arc. So the proof is valid in this case, too. If we denote by $C(E)$ the logarithmic capacity of the arc E and pose $w_N = (z - z_1)(z - z_2) \cdots (z - z_N)$, then the following statement holds:

LEMMA 2.

$$\mu^*(\varrho) = \mu\left(\varrho \frac{|w_N|^2}{C(E)^{2N}} \right),$$

where the right-hand member is the extremal constant of problem (2) *associated with the weight function* $\varrho(\zeta)\, |w_N(\zeta)|^2 / C(E)^{2N}$.

Proof. This lemma is proved for a curve in [4]. The proof is based on the fact that if $f(z) \in H^2(\Omega, \rho)$ and $f(z_k) = 0$, $f(\infty) = 1$, then $\psi(z) = f(z)\, C(E)^N\, \Phi^N(z)/w_N(z)$ is from the same space and $\psi(\infty) = 1$. It is true for an arc. So the proof is valid for an arc.

We denote by $m_{n,2}(\sigma)$ the extremal constants associated with the measure σ,

$$m_{n,2}(\sigma) := \min\{ \|Q_n(z)\|^2_{L^2(\sigma, F)}, Q_n(z) = z^n + \cdots \}, \tag{5}$$

where $F = E \cup \{z_1, z_2, ..., z_N\}$. The polynomials $T_{n,2}(\sigma)$ that are orthogonal with respect to the measure σ are extremal polynomials for problem (5). Finally, we remark that the Szegő function $D(z)$ has a reproducing property: for $K(z, \infty) := \phi^*(z)/\mu(\varrho) = D(z)\, D^2(\infty)/2\pi$ and for all $\phi(z)$ from $H^2(\Omega, \varrho)$, we have

$$\phi(\infty) = \oint_E \phi(\zeta)\, \overline{K(\zeta, \infty)}\, \rho(\zeta)\, |d\zeta|$$

(see [10, p. 173]).

We say, following Widom, that an arc E is from the $C^{\alpha+}$ class if E is rectifiable and its coordinates are α-times differentiable, with αth derivatives satisfying a Lipschitz condition with some positive exponent.

3. Asymptotics of the Orthogonal Polynomials

The result of this Note is:

THEOREM 1. *Suppose that* $E \in C^{2+}$ *and that* $\varrho(\zeta)$ *satisfies Szegő condition* (1). *Then we have*

1. $\lim m_{n,2}(\sigma)/C(E)^{2n} = \mu^*(\rho)$,
2. $\int_E |C(E)^{-n} T_{n,2}(\zeta) - H_n(\zeta)|^2 \rho(\zeta)\,|d\zeta| \to 0$,
3. $T_{n,2}(z) = C(E)^n \Phi^n(z)[\psi^*(z) + \varepsilon_n(z)]$,

where $H_n(\zeta) = \Phi_+^n(\zeta)\,\psi_+^*(\zeta) + \Phi_-^n(\zeta)\,\psi_-^*(\zeta)$, $\varepsilon_n \to 0$ *uniformly on the closed subsets of* Ω.

Proof. We set $\phi_n^*(z) = T_{n,2}(z)/C(E)^n\,\Phi^n(z)$. This function is from $H^2(\Omega, \varrho)$ space, it has boundary values on both sides of the arc E, and

$$2m_{n,2}(\sigma)/C(E)^{2n} = \|\phi_n^*\|^2_{H^2(\Omega,\rho)} + \sum_1^N A_k\,|\phi_n^*(z_k)|^2\,|\Phi^{2n}(z_k)| \tag{6}$$

(for the norm of ϕ in $H^2(\Omega, \rho)$ we take the integral on the two sides of E). The extremal property of ϕ_n^* implies (see [4] for the details) that

$$\|\phi_n^*\|^2_{H^2(\Omega,\rho)} \leqslant 2m_{n,2}(\sigma)/C(E)^{2n} \leqslant \|\psi_{n-N}^*\|^2_{H^2(\Omega,\rho^*)}, \tag{7}$$

where $\varrho^* = \varrho\,|w_N|^2/C(E)^{2N}$ and ψ_n^* is the extremal function associated with the $m_{n,2}(\varrho^*)$ extremal problem. From Widom's theorem [10, p. 219], we have

$$\|\psi_{n-N}^*\|_{H^2(\Omega,\varrho^*)} \to 2\mu(\varrho^*). \tag{8}$$

This implies that

$$\limsup \|\phi_n^*\|^2_{H^2(\Omega,\varrho)} \leqslant 2\mu(\varrho^*) = 2\mu^*(\varrho)$$

(see Lemma 2). Equations (6) and (7) imply that the products $|\phi_n^*(z_k)|^2\,|\Phi^{2n}(z_k)|$ are bounded for all $k = 1, 2, ..., N$. So $\phi_n^*(z_k) \to 0$, $n \to \infty$, ($|\Phi(z_k)| > 1$). This is a starting point for what follows.

Now we take the integral

$$I_n = \int_E |C(E)^{-n}\,T_{n,2}(\zeta) - H_n(\zeta)|^2\,\varrho(\zeta)\,|d\zeta|$$

and transform it as usual in the following sum:

$$I_n = \int_E |C(E)^{-n} T_{n,2}(\zeta)|^2 \varrho(\zeta)\, |d\zeta| + \int_E |H_n(\zeta)|^2 \varrho(\zeta)\, |d\zeta|$$

$$-2\,\mathrm{Re} \int_E C(E)^{-n} T_{n,2}(\zeta)\, \overline{H_n(\zeta)}\, \varrho(\zeta)\, |d\zeta|.$$

Then

$$\int_E |C(E)^{-n} T_{n,2}(\zeta)|^2 \varrho(\zeta)\, |d\zeta| = \tfrac{1}{2} \|\phi_n^*\|^2_{H^2(\Omega,\varrho)} \leqslant \mu^*(\varrho) + \alpha_n, \qquad \alpha_n \to 0,$$

and

$$\int_E |H_n(\zeta)|^2 \rho(\zeta)\, |d\zeta|$$

$$= \|\psi^*\|^2_{H^2(\Omega,\rho)} + 2\,\mathrm{Re} \int_E \overline{\Phi^n_+(\zeta)\, \psi^*_+(\zeta)}\, \Phi^n_-(\zeta)\, \psi^*_-(\zeta)\, \rho(\zeta)\, |d\zeta|.$$

The second term approaches 0 if $n \to \infty$ (this is Widom's lemma (see [10, p. 128]), analogous to the Riemann–Lebesgue lemma for an integrable functions). So

$$\int_E |H_n(\zeta)|^2 \rho(\zeta)\, |d\zeta| = \mu^*(\rho) + \beta_n, \qquad \beta_n \to 0.$$

Now, we use the relations $\psi^*(z) = \phi^*(z)\, B(z)$, $\phi^*(z) = \mu(\rho)\, K(z, \infty)$, K being the reproducing kernel, $\mu^*(\rho) = (\prod_1^N |\Phi(z_k)|)^2\, \mu(\rho)$, and $\overline{\Phi_+} = 1/\Phi_+$, $\zeta \in E$, for the following tranformations:

$$2\,\mathrm{Re} \int_E C^{-N}(E)\, T_{n,2}(\zeta)\, \overline{H_n(\zeta)}\, \rho(\zeta)\, |d\zeta|$$

$$= 2\,\mathrm{Re} \oint_E \frac{T_{n,2}(\zeta)}{C^N(E)\, \Phi^n(\zeta)}\, \overline{\psi^*(\zeta)}\, \rho(\zeta)\, |d\zeta|$$

$$= 2\,\mathrm{Re} \oint_E \frac{T_{n,2}(\zeta)}{C^N(E)\, \Phi^n(\zeta)}\, \mu(\rho)\, \overline{B(\zeta)\, K(\zeta, \infty)}\, \rho(\zeta)\, |d\zeta|$$

$$= 2\mu(\rho)\, \mathrm{Re} \oint_E \frac{\phi_n^*(\zeta)}{B(\zeta)}\, |B(\zeta)|^2\, \overline{K(\zeta, \infty)}\, \rho(\zeta)\, |d\zeta|$$

$$= 2\mu^*(\rho)\, \mathrm{Re} \oint_E \frac{\phi_n^*(\zeta)}{B(\zeta)}\, \overline{K(\zeta, \infty)}\, \rho(\zeta)\, |d\zeta|.$$

We set $\psi_n = \phi_n^*/B$; then

$$\psi_n = \sum_1^N \frac{c_k \phi_n^*(z_k)}{z - z_k} + r_n(z),$$

where $c_k = 1/B'(z_k)$. But $\phi_n^*(z_k) \to 0$ and $r_n(\infty) = 1$, $r_n \in H^2(\Omega, \rho)$, this and the reproducing properties of K imply that

$$2 \operatorname{Re} \int_E C^{-N}(E)\, T_{n,2}(\zeta)\, \overline{H_n(\zeta)}\, \rho(\zeta)\, |d\zeta| = 2\mu^*(\rho) + \delta_n, \qquad \delta_n \to 0.$$

Finally,

$$I_n = \int_E |C(E)^{-n}\, T_{n,2}(\zeta) - H_n(\zeta)|^2\, \varrho(\zeta)\, |d\zeta|$$
$$\leqslant \mu^*(\rho) + \alpha_n + \mu^*(\rho) + \beta_n - 2\mu^*(\rho) - \delta_n.$$

The asymptotic of the extremal constants $m_{n,2}(\sigma)$ follows immediately from this. For the asymptotics in the region exterior to the arc E, we need the Szegő reproducing kernel function $K(\zeta, z)$ (see [10, p. 173]): $\forall \phi \in H^2(\Omega, \rho)$, $\forall z \in \Omega$,

$$\phi(z) = \oint_E \phi(\zeta)\, \overline{K(\zeta, z)}\, \rho(\zeta)\, |d\zeta|;$$

then

$$\frac{T_{n,2}(z)}{C^n(E)\, \Phi^n(z)}$$
$$= \oint_E \frac{T_{n,2}(\zeta)}{C^n(E)\, \Phi^n(\zeta)}\, \overline{K(\zeta, z)}\, \rho(\zeta)\, |d\zeta|$$
$$= \int_E C^{-n}(E)\, T_{n,2}(\zeta) \{\Phi_+^{-n}(\zeta)\, \overline{K_+(\zeta, z)} + \Phi_-^{-n}(\zeta)\, \overline{K_-(\zeta, z)}\}\, \rho(\zeta)\, |d\zeta|$$
$$= \int_E \{C^{-n}(E)\, T_{n,2}(\zeta) - H_n(\zeta)\}$$
$$\times \{\Phi_+^{-n}(\zeta)\, \overline{K_+(\zeta, z)} + \Phi_-^{-n}(\zeta)\, \overline{K_-(\zeta, z)}\}\, \rho(\zeta)\, |d\zeta|$$
$$+ \int_E H_n(\zeta) \{\Phi_+^{-n}(\zeta)\, \overline{K_+(\zeta, z)} + \Phi_-^{-n}(\zeta)\, \overline{K_-(\zeta, z)}\}\, \rho(\zeta)\, |d\zeta|.$$

The first integral approaches 0 as $n \to \infty$ (part 2 of the theorem); the second one may be transformed into the form

$$\begin{aligned}
&\int_E \{\Phi_+^{-n}(\zeta)\,\psi_+^*(\zeta) + \Phi_-^{-n}(\zeta)\,\psi_-^*(\zeta)\} \\
&\qquad \times \{\Phi_+^{-n}(\zeta)\,\overline{K_+(\zeta, z)} + \Phi_-^{-n}(\zeta)\,\overline{K_-(\zeta, z)}\}\,\rho(\zeta)\,|d\zeta| \\
&\quad = \oint_E \psi^*(\zeta)\,\overline{K(\zeta, z)}\,\rho(\zeta)\,|d\zeta| \\
&\qquad + \int_E \{\Phi_+^n(\zeta)\,\psi_+^*(\zeta)\,\Phi_-^{-n}(\zeta)\,\overline{K_-(\zeta, z)} \\
&\qquad + \psi_-^*(\zeta)\,\Phi_+^{-n}(\zeta)\,\Phi_-^n(\zeta)\,\overline{K_+(\zeta, z)}\}\,\rho(\zeta)\,|d\zeta| \\
&\quad = \phi^*(\zeta) + \lambda_n,
\end{aligned}$$

where $\lambda_n \to 0$ (Fourier coefficients of an integrable function). This proves part 3.

Remark. Part $2 \Rightarrow 3$ is not correct in Widom's proof [10, p. 222] for the case of an arc; the present proof is valid for this case, too. The theorem is proved.

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